FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 2
HSC ASSESSMENT TASK 3, 2024
ANSWERS COVER SHEET

Student Number:

_____

QUESTION	MARK	MEX 12-1	MEX 12-2	MEX 12-3	MEX 12-4	MEX 12-5	MEX 12-6	MEX 12-7	MEX 12-8
Q1 – Q3	/3					✔		✔	✔
Q4 - Q5	/2			✔				✔	✔
Q6	/14					✔		✔	✔
Q7	/12					✔		✔	✔
Q8	/12					✔		✔	✔
Q9	/13			✔				✔	✔
Q10	/15			✔				✔	✔
Q11	/14			✔				✔	✔
TOTAL	/85			/44		/41		/85	/85

GIRRAWEEN HIGH SCHOOL

MATHEMATICS

EXTENSION 2, 2024

TASK 3

Time allowed – 90 minutes

Instructions:

- There are 11 questions in this paper. All questions ae compulsory.
- Use blue or black pen.
- Write all your answers in the answer booklets provided.
- For Questions 1-5, fill in the circle corresponding to the correct answer in your answer booklet.
- For Questions 6-11, start each question on a new page.
- Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- NESA – approved calculators may be used.
- Mathematics reference sheets are provided.

Questions 1-5 (5 marks)

Colour in the circle corresponding to the correct answer on answer sheet provided in your answer booklet.

1. Which of the following, for $x > 0$, is an expression for $\int \frac{1}{x^3 + x}\, dx$?

(A) $\log_e\left(x\sqrt{x^2+1}\right)+C$

(B) $\log_e\left(x(x^2+1)\right)+C$

(C) $\log_e\left(\frac{x}{\sqrt{x^2+1}}\right)+C$

(D) $\log_e\left(\frac{x}{x^2+1}\right)+C$

2. With a suitable substitution, $\int_0^{\frac{\pi}{6}} \sin^3(2x)dx$ can be expressed as

(A) $2\int_1^{\frac{1}{2}} (u^2 - 1)du$

(B) $\frac{1}{2}\int_1^{\frac{1}{2}} (u^2 - 1)du$

(C) $2\int_1^{\frac{1}{2}} (1-u^2)du$

(D) $\frac{1}{2}\int_1^{\frac{1}{2}} (1-u^2)du$

Please turn over

3. If $f(x) = f(a-x)$, then $\int_0^a xf(x)dx =$

(A) $\int_0^a f(x)dx$

(B) $\int_0^a f(a-x)dx$

(C) $\frac{a}{2}\int_0^a f(x)dx$

(D) $-\frac{a}{2}\int_0^a f(x)dx$

4. What is the distance of the point (2, 3, 7) from the $y-z$ plane?

(A) 2 units

(B) 3 units

(C) 7 units

(D) 9 units

5. P, Q and R are three collinear points with position vectors $\underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively, where Q lies between P and R. If $\left|\overrightarrow{QR}\right| = \frac{1}{2}\left|\overrightarrow{PQ}\right|$, then $\underset{\sim}{r} =$

(A) $\frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

(B) $\frac{1}{2}\underset{\sim}{q} - \frac{3}{2}\underset{\sim}{p}$

(C) $\frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

(D) $\frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

Please turn over

Question 6 (14 marks) start each question on a new page. Marks

Evaluate:

(a) $\int \sin^4 x \cos^3 x dx$ 3

(b) $\int x \sec^2 x dx$ 3

(c) $\int \frac{x^2+1}{x^2+2x+4} dx$ 4

(d) Use the substitution $t = \tan\frac{x}{2}$ to show that $\int_0^{\frac{\pi}{3}} \frac{1}{8\sin x + 6\cos x - 10} dx = \frac{1}{2\left(2-\sqrt{3}\right)}$ 4

Question 7 (12 marks) start each question on a new page.

(a) (i) Find real numbers A, B and C such that

$$\frac{16x-43}{(x+2)(x-3)^2} \equiv \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$ 3

(ii) Hence evaluate $\int \frac{16x-43}{(x+2)(x-3)^2} dx$ 3

(b) (i) Given that $I_n = \int_1^e x^3 (\log_e x)^n dx$, where $n \geq 0$ is an integer.

Show that $I_n = \frac{e^4}{4} - \frac{n}{4} I_{n-1}$ 3

(ii) Hence show that $I_2 = \frac{5e^4}{32} - \frac{1}{32}$. 3

Please turn over

Question 8 (12 marks) start each question on a new page.

(a) (i) Show that $\tan\left(\frac{\pi}{4}-x\right)=\frac{1-\tan x}{1+\tan x}$ 1

(ii) Using the result in **(i)** and the property $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, evaluate

$$\int_0^{\frac{\pi}{4}} \log_e(1+\tan x)\,dx$$ 3

(b) (i) If $I_n=\int \cot^n x\,dx$, prove that $I_n=-\frac{1}{n-1}\cot^{n-1}x-I_{n-2}$ 4

(ii) Hence evaluate $\int \cot^5 x\,dx$ 4

Question 9 (13 marks) start each question on a new page.

(a) Write the vector equation of the line segment joining $(3,-2,3)$ and $(5,-3,4)$ 2

(b) Find the coordinates of the point P that divides $A(2,-1,4)$ and $B(4,3,2)$ internally in the ratio 2:3. 3

(c) Let P, A and B be the points $(-4,3,-1),(3,2,1)$ and $(0,-4,1)$ respectively.

(i) Find $\overrightarrow{AP}$ and $\overrightarrow{AB}$. 2

(ii) Find $proj_{\underset{\sim}{b}}\underset{\sim}{p}$ where $\underset{\sim}{p}=\overrightarrow{AP}$ and $\underset{\sim}{b}=\overrightarrow{AB}$. 4

(iii) Hence calculate the perpendicular distance d from the point P to the line AB. 2

Please turn over

Question 10 (15 marks) start each question on a new page.

(a) (i) Evaluate the direction cosines for $\underset{\sim}{a} = \overrightarrow{OA}$ where A is the point $(4,-2,5)$ 3

(ii) Hence calculate $\angle AOX$ to the nearest degree. 1

(b) Find the centre and radius of the sphere $x^2 + y^2 + z^2 - 6x + 4y + 2z - 11 = 0$. 4

(c) Given the points $A(1,3,-2)$ and $B(2,-1,5)$

(i) Evaluate $\overrightarrow{AB}$. 1

(ii) Hence find the vector equation of the line AB. 2

(iii) Determine if the point $C(3,4,9)$ lies on the line. 1

(iv) Consider a line with parametric equations $x = 1-\mu, y = 2+3\mu, z = -1+\mu$. Assuming this line is neither parallel nor perpendicular to AB, determine whether the lines intersect or are skew. 3

Please turn over

Question 11(14 marks) start each question on a new page.

(a) (i) Find the point of intersection of the lines

$$\underset{\sim}{r} = \begin{pmatrix} 8 \\ 16 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \text{ and } \underset{\sim}{v} = \begin{pmatrix} 14 \\ 12 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} 12 \\ 8 \\ 10 \end{pmatrix}$$ 4

(ii) Find the acute angle between the lines (to the nearest degree) 3

(b) (i) Write the vector equation of the sphere $(x+2)^2 + (y-3)^2 + (z+4)^2 = 125$ 1

(ii) Find the points of intersection of the sphere given in **(i)** with the line

$$\underset{\sim}{r} = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 13 \\ -11 \end{pmatrix}$$ 4

(c) A function is defined by the vector equation $\underset{\sim}{r} = (2+\sin\theta)\underset{\sim}{i} + (1-\cos\theta)\underset{\sim}{j}, \theta \geq 0$.

Find the Cartesian equation of the function. 2

END OF EXAMINATION

Year 12 Extension 2 Task 3 2024 Solutions

Multiple choice (5 marks)

1. $\int \frac{1}{x^3+x}\,dx$ (C)

$= \int \frac{1}{x(x^2+1)}\,dx$ $\qquad \frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$

$= \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$

$= \log_e x - \frac{1}{2}\log(x^2+1) + C$

$= \log_e \frac{x}{\sqrt{x^2+1}} + C$

2. $\int_0^{\frac{\pi}{6}} \sin^3(2x)\,dx$ (B)

$= \int_0^{\frac{\pi}{6}} (1-\cos^2 2x)\sin 2x\,dx$

$u = \cos 2x$

$\frac{du}{dx} = -2\sin 2x$

$\sin 2x\,dx = -\frac{du}{2}$

when $x = 0$, $u = 1$

when $x = \frac{\pi}{6}$, $u = \frac{1}{2}$

$= \int_1^{\frac{1}{2}} (1-u^2)\times -\frac{du}{2}$

$= \frac{1}{2}\int_1^{\frac{1}{2}} (u^2-1)\,du$

3. $\int_0^a x f(x)\,dx$ (C)

$= \int_0^a (a-x) f(a-x)\,dx$

$= \int_0^a (a-x) f(x)\,dx$

$= \int_0^a a f(x)\,dx - \int_0^a x f(x)\,dx$

$2\int_0^a x f(x)\,dx = \int_0^a a f(x)\,dx$

$\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$

4. (A)

5. $\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$ (D)

$\underset{\sim}{r} - \underset{\sim}{q} = \frac{1}{2}(\underset{\sim}{q} - \underset{\sim}{p})$

$\underset{\sim}{r} = \frac{1}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p} + \underset{\sim}{q}$

$= \frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

Question 6 (14 marks)

(a) $\int \sin^4 x \cos^3 x \, dx$ (3 marks)

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$= \int \sin^4 x \cos^2 x \cdot \cos x \, dx$

$= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx$

$= \int (u^4 - u^6) \, du$

$= \frac{u^5}{5} - \frac{u^7}{7} + C$

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

(b) $\int x \sec^2 x \, dx$ (3 marks)

$= x \tan x - \int 1 \times \tan x \, dx$

$= x \tan x - \int \frac{\sin x}{\cos x} \, dx$

$= x \tan x - \int \frac{-\sin x}{-\cos x} \, dx$

$= x \tan x + \int \frac{-\sin x}{\cos x} \, dx$

$= x \tan x + \log |\cos x| + C$

Very well done. Students have shown good understanding regarding the technique to integrate both (a) and (b)

Only very few wrote $-\log|\cos x|$ instead of $+\log|\cos x|$ watch out for silly mistakes.

(c) $\int \frac{x^2+1}{x^2+2x+4}\,dx$ (4 marks)

$$x^2+2x+4 \overline{)\,x^2+1\,(\,1}$$
$$\frac{x^2+2x+4}{-2x-3}$$

$$x^2+1 = (x^2+2x+4)-(2x+3)$$

$$\frac{x^2+1}{x^2+2x+4} = 1 - \frac{2x+3}{x^2+2x+4}$$

$$I = \int \left[1 - \frac{2x+3}{x^2+2x+4}\right] dx$$

$$= \int dx - \int \frac{2x+3}{x^2+2x+4}\,dx$$

$$= \int dx - \int \frac{2x+2+1}{x^2+2x+4}\,dx$$

$$= \int dx - \int \frac{2x+2}{x^2+2x+4}\,dx - \int \frac{dx}{x^2+2x+4}$$

$$= x - \log|x^2+2x+4| - I_1$$

$$I_1 = \int \frac{dx}{(x+1)^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

$$I = x - \log|x^2+2x+4| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

Students have shown good understanding of using partial fractions in integration.

Most students have shown understanding of reduction formulae

(d) $\int_0^{\frac{\pi}{3}} \frac{1}{8\sin x + 6\cos x - 10}\, dx$

(4 marks)

Let $t = \tan\frac{x}{2}$, $\frac{dt}{dx} = \frac{\sec^2\frac{x}{2}}{2}$

$dt = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)dx$

$dt = \frac{1}{2}(1+t^2)\,dx$

$dx = \frac{2\,dt}{1+t^2}$

$8\sin x + 6\cos x - 10 = 8\times\frac{2t}{1+t^2} + \frac{6(1-t^2)}{1+t^2} - 10$

$= \frac{-16t^2 + 16t - 4}{1+t^2}$

$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{-16t^2+16t-4} \times \frac{2\,dt}{1+t^2}$

$= -\frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{4t^2-4t+1} = -\frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}} (2t-1)^{-2}\,dt$

$= \frac{1}{4}\left[\frac{1}{(2t-1)}\right]_0^{\frac{1}{\sqrt{3}}} = \frac{1}{4}\left(\frac{1}{\frac{2-\sqrt{3}}{\sqrt{3}}} + 1\right)$

$= \frac{1}{4}\left(\frac{\sqrt{3}}{2-\sqrt{3}} + 1\right) = \frac{1}{4}\times\frac{2}{2-\sqrt{3}}$

$= \frac{1}{2(2-\sqrt{3})}$

Most students made mistakes in simplify the algebraic expressions.

Question 7 (12 marks)

(a)(i) $\dfrac{16x-43}{(x+2)(x-3)^2} = \dfrac{A}{x+2} + \dfrac{B}{x-3} + \dfrac{C}{(x-3)^2}$ (3 marks)

$16x-43 = A(x-3)^2 + B(x+2)(x-3) + C(x+2)$

Substitute $x=3$

$48-43 = 5C$

$\underline{C=1}$

Substitute $x=-2$

$-32-43 = A(-5)^2$

$\underline{A=-3}$

Equating coefficients of x^2

$A+B=0$

$\underline{B=3}$

Whole question was generally very well done.

(ii) $\displaystyle\int \frac{16x-43}{(x+2)(x-3)^2}\,dx$ (3 marks)

$= \displaystyle\int \left(\frac{-3}{x+2} + \frac{3}{x-3} + \frac{1}{(x-3)^2}\right)dx$

$= -3\log|x+2| + 3\log|x-3| - \dfrac{1}{x-3} + C$

$= 3\left[\log|x-3| - \log|x+2|\right] - \dfrac{1}{x-3} + C$

$= 3\log\dfrac{|x-3|}{|x+2|} - \dfrac{1}{x-3} + C$

(b)(i) $I_n = \int_1^e x^3 (\log_e x)^n \, dx$

(3 marks)

$= (\log_e x)^n \times \frac{x^4}{4} \Big]_1^e - \int n (\log_e x)^{n-1} \times \frac{1}{x} \times \frac{x^4}{4} \, dx$

$= (\log_e e)^n \times \frac{e^4}{4} - 0 - \frac{n}{4} \int x^3 (\log_e x)^{n-1} \, dx$

$= \frac{e^4}{4} - \frac{n}{4} \int x^3 (\log_e x)^{n-1} \, dx$

$I_n = \frac{e^4}{4} - \frac{n}{4} I_{n-1}$

Whole question was generally very well done.

(ii) $I_2 = \frac{e^4}{4} - \frac{2}{4} I_1$

(3 marks)

$I_1 = \frac{e^4}{4} - \frac{1}{4} I_0$

$I_0 = \int_1^e x^3 \, dx = \left[\frac{x^4}{4}\right]_1^e = \frac{1}{4}(e^4 - 1)$

$I_2 = \frac{e^4}{4} - \frac{2}{4}\left[\frac{e^4}{4} - \frac{1}{4} I_0\right]$

$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8} \times \frac{1}{4}(e^4 - 1)$

$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{3} e^4 - \frac{1}{32}$

$= \frac{5e^4}{32} - \frac{1}{32}$

Question 8 (12 marks)

(a)(i) (1 marks)

$$\tan\left(\frac{\pi}{4}-x\right) = \frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}$$

$$= \frac{1-\tan x}{1+\tan x}$$

Some students used the formula $\tan(A+B)$ and got the correct answer.

(ii) (3 marks)

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\log 2 - \log(1+\tan x)\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2\,dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$$

$$I = \log 2 \int_0^{\frac{\pi}{4}} dx - I$$

$$2I = \log 2\,[x]_0^{\frac{\pi}{4}} = \log 2\left(\frac{\pi}{4}-0\right)$$

$$= \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8}\log 2$$

Majority of students did this question in a different way which is harder than this method.

Forgot to divide by 2.

(b)(i) $I_n = \int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx$

(4 marks)

$= \int \cot^{n-2} x \left(\operatorname{cosec}^2 x - 1\right) dx$

Some wrote $1 - \operatorname{cosec}^2 x$ instead of $\operatorname{cosec}^2 x - 1$

$= \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$

Let $u = \cot x$

$\frac{du}{dx} = -\operatorname{cosec}^2 x$

$I_n = \int u^{n-2} x - du - I_{n-2}$

$= -\int u^{n-2} du - I_{n-2}$

$= -\frac{u^{n-1}}{n-1} - I_{n-2} = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$

(ii) $I_5 = \frac{-1}{4} \cot^4 x - I_3$

(4 marks)

$I_3 = \frac{-1}{2} \cot^2 x - I_1$

$I_1 = \int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \log|\sin x| + C$

done well.

$I_5 = \frac{-1}{4} \cot^4 x + \frac{1}{2} \cot^2 x + I_1$

$= \frac{-1}{4} \cot^4 x + \frac{1}{2} \cot^2 x + \log|\sin x| + C$

Question 9 (13 marks)

(a) (2 marks)

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5-3 \\ -3+2 \\ 4-3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad 0 \leq \lambda \leq 1$$

Mostly done well.

(b) (3 marks)

$$\frac{AP}{PB} = \frac{2}{3}$$

$$3AP = 2PB$$

$$3\begin{pmatrix} x-2 \\ y+1 \\ z-4 \end{pmatrix} = 2\begin{pmatrix} 4-x \\ 3-y \\ 2-z \end{pmatrix}$$

$$3(x-2) = 2(4-x)$$

$$3x-6 = 8-2x$$

$$5x = 14$$

$$x = \frac{14}{5}$$

$$3(y+1) = 2(3-y)$$

$$3y+3 = 6-2y$$

$$5y = 3$$

$$y = \frac{3}{5}$$

$$3(z-4) = 2(2-z)$$

$$5z = 16$$

$$z = \frac{16}{5}$$

$$P\begin{pmatrix} \frac{14}{5} \\ \frac{3}{5} \\ \frac{16}{5} \end{pmatrix}$$

Most students have shown good understanding of this concept. Students are advised to revise this concept thoroughly.

(c)(i) $P(-4,3,-1)$ $A(3,2,1)$ $B(0,-4,1)$

(2 marks) $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= (-4-3, 3-2, -1-1) = \underline{(-7,1,-2)}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \underline{(-3,-6,0)}$

(ii) $\text{Proj}_{\underset{\sim}{b}} \underset{\sim}{p} = \dfrac{\underset{\sim}{p} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \cdot \underset{\sim}{b}$

(4 marks)

$\underset{\sim}{p} = \begin{pmatrix} -7 \\ 1 \\ -2 \end{pmatrix} \quad \underset{\sim}{b} = \begin{pmatrix} -3 \\ -6 \\ 0 \end{pmatrix}$

$\underset{\sim}{p} \cdot \underset{\sim}{b} = (-7 \times -3) + (1 \times -6) + 0 = 15$

$\underset{\sim}{b} \cdot \underset{\sim}{b} = 9 + 36 = 45$

$\text{Proj}_{\underset{\sim}{b}} \underset{\sim}{p} = \dfrac{15}{45}(-3\underset{\sim}{i} - 6\underset{\sim}{j})$

$= \dfrac{1}{3}(-3\underset{\sim}{i} - 6\underset{\sim}{j})$

$= \underline{\underset{\sim}{i} - 2\underset{\sim}{j}}$

Mostly done well

→ some students quoted the wrong formula and hence lost marks.

(iii) perpendicular distance (2 marks)

$$= \left| \text{Proj}_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p} \right|$$

$$= \left| (\underset{\sim}{i} - 2\underset{\sim}{j}) - (-7\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) \right|$$

$$= \left| 6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k} \right|$$

$$= \sqrt{36+9+4} = \sqrt{49} = 7 \text{ units.}$$

Most students who have done (ii) correctly were successful in doing this part correctly. Most students have demonstrated that they have a clear understanding of finding a distance

Question 10 (15 marks)

(a)(i) $|\underset{\sim}{a}| = \sqrt{16+4+25} = \sqrt{45}$

(3 marks)

direction cosines are

$$\frac{4}{\sqrt{45}}, \frac{-2}{\sqrt{45}} \text{ and } \frac{5}{\sqrt{45}}$$

(ii) $\cos \angle AOX = \frac{4}{\sqrt{45}}$

(1 mark)

$$\angle AOX = 53^0$$

(b) $x^2 - 6x + y^2 + 4y + z^2 + 2z = 11$

$x^2 - 6x + 9 + y^2 + 4y + 4 + z^2 + 2z + 1 = 11 + 9 + 4 + 1$

(4 marks) $(x-3)^2 + (y+2)^2 + (z+1)^2 = 25$

Centre $(3, -2, -1)$ radius $= 5$

(c)(i) $A = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ $B \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$

(1 mark)

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$$

$$= \begin{pmatrix} 2-1 \\ -1-3 \\ 5--2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$$

(ii) Equation of the line $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

(2 marks)

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$$

Q.(10) generally very well done, although students need to remember that direction COSINES are just COSINES, NOT the actual ANGLES.

(iii) (1 mark)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ 3-4\lambda \\ -2+7\lambda \end{pmatrix}$$

Substitute $\begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}$

$1+\lambda = 3$ $\quad 3-4\lambda = 4$ $\quad -2+7\lambda = 9$

$\lambda = 2$ $\quad \lambda = -\frac{1}{4}$ $\quad \lambda = \frac{11}{7}$

Since the values of λ are different, the point C does not lie on the line.

Generally well done.

(iv) (3 marks) Equating the parametric equations of the two lines

$$1+\lambda = 1-\mu$$
$$\lambda+\mu = 0$$

$$3-4\lambda = 2+3\mu$$
$$4\lambda+3\mu = 1$$

$$7\lambda-2 = \mu-1$$
$$7\lambda-\mu = 1$$

$\lambda + \mu = 0$ —— (1)

$4\lambda + 3\mu = 1$ —— (2)

$7\lambda - \mu = 1$ —— (3)

(1) × 4 $\quad 4\lambda + 4\mu = 0$

$4\lambda + 3\mu = 1$

$\mu = -1$

$\lambda = -\mu = -(-1) = 1$

Substitute $\lambda = 1$ and $\mu = -1$ in (3)

LHS $= 7\lambda - \mu$

$= 7 \times 1 - (-1) = 8$

RHS $= 1$ $\quad$ LHS $\neq$ RHS

The lines do not intersect and are therefore skew.

Generally well done.

Question 11 (14 marks)

(a)(i) (4 marks)

$$\begin{pmatrix} 8 \\ 16 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ 12 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} 12 \\ 8 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 8+2\lambda \\ 16+4\lambda \\ 6+2\lambda \end{pmatrix} = \begin{pmatrix} 14+12\mu \\ 12+8\mu \\ 10+10\mu \end{pmatrix}$$

$8+2\lambda = 14+12\mu$

$16+4\lambda = 12+8\mu$

$6+2\lambda = 10+10\mu$

$\lambda - 6\mu = 3$ — (1)

$\lambda - 2\mu = -1$ — (2)

$\lambda - 5\mu = 2$ — (3)

(1) − (2) $-4\mu = 4$

$\mu = -1$

$\lambda = 3 + 6\mu$

$= 3 - 6 = -3$

Some students didn't substitute the values of λ and μ into the unused equation.

Substitute $\lambda = -3$, $\mu = -1$ in (3)

LHS $= \lambda - 5\mu = 2$, RHS $= 2$

Point of intersection is

$$\begin{pmatrix} 8+2\lambda \\ 16+4\lambda \\ 6+2\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

(ii) Let θ be the angle between the lines

(3 marks)

$$\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 8 \\ 10 \end{pmatrix} = \left|\left| \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \right|\right| \left|\left| \begin{pmatrix} 12 \\ 8 \\ 10 \end{pmatrix} \right|\right| \cos\theta$$

$$(2\times12)+(4\times8)+(2\times10) = \sqrt{4+16+4} \cdot \sqrt{144+64+100}\cos\theta$$

$$76 = \sqrt{24}\ \sqrt{308}\ \cos\theta$$

$$\cos\theta = \frac{76}{\sqrt{7392}}$$

$$\theta = 28^\circ$$

Some algebraic mistakes were made by some students.

(b)(i) (1 mark) $\left| \underset{\sim}{v} - \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix} \right| = \sqrt{125}$

done well

(ii) (4 marks) $\underset{\sim}{r} = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 13 \\ -11 \end{pmatrix}$

$= \begin{pmatrix} 4+2\lambda \\ -5+13\lambda \\ 1-11\lambda \end{pmatrix}$

Substitute $x = 4+2\lambda$, $y = -5+13\lambda$

and $z = 1-11\lambda$ in

$(x+2)^2 + (y-3)^2 + (z+4)^2 = 125$

$(6+2\lambda)^2 + (13\lambda-8)^2 + (5-11\lambda)^2 = 125$

$36 + 4\lambda^2 + 24\lambda + 169\lambda^2 + 64 - 208\lambda$

$+ 25 - 110\lambda + 121\lambda^2 = 125$

$294\lambda^2 - 294\lambda + 125 = 125$

$294\lambda^2 - 294\lambda = 0$

$294\lambda(\lambda - 1) = 0$

$\lambda = 0$ or $\lambda = 1$

Few made mistake in expansion and got wrong quadratic

For $\lambda = 0$

$$\begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$$

For $\lambda = 1$

$$\begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 13 \\ -11 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 8 \\ -10 \end{pmatrix}$$

(c) $\underset{\sim}{r} = (2+\sin\theta)\underset{\sim}{i} + (1-\cos\theta)\underset{\sim}{j}$

(2 marks)

$x = 2+\sin\theta$

$y = 1-\cos\theta$

$\sin\theta = x-2$

$\cos\theta = 1-y$

$\sin^2\theta + \cos^2\theta = 1$

$(x-2)^2 + (1-y)^2 = 1$

simple question
But many have no idea how to proceed.
This is just conversion between parametric and cartesian equation.